Limit, Continuity, Differentiation and Integration

LIMIT OF A FUNCTION

Definition: Let f(x) be a function defined on R. $\lim_{x\to\infty} f(x) = \ell$ means that for any ε > 0 , there exists X > 0 such that when x > X , $|f(x) - \ell| < \varepsilon$.

Note: (1) $\lim_{x\to\infty} f(x) = \ell$ means that the difference between f(x) and A can be made arbitrarily small when x is sufficiently large.

(2) $\lim_{x\to\infty} f(x) = \ell$ means $f(x) \to A$ as $x \to \infty$. (3) Infinity, ∞, is a symbol but not a real value

Limit of a Function at a Point

Definition: Let f(x) be a function defined on R. $\lim_{x\to a} f(x) = \ell$ means that for any ε > 0 , there exists δ > 0 such that when $0 < |x - a| < \delta$, $|f(x) - \ell| < \varepsilon$.

Note: (1) $\lim_{x\to a} f(x) = \ell$ means that the difference between f(x) and A can be made arbitrarily small when x is sufficiently close to a.

(2) If *f(x)* is a polynomial , then $\lim_{x\to a} f(x) = f(a)$.

(3) In general , $\lim_{x\to a} f(x) \neq f(a)$.

(4) f(x) may not defined at x = a even through $\lim_{x\to a} f(x)$ exists.

EXISTANCE OF THE LIMIT:

Limit of a function is said to be exist if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ = finite quantity

Where $\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(x+h)$ and $\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(x-h)$

Rules of Operations on Limits: If $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist , then

(a) $\lim_{x\to\infty}[f(x) \pm g(x)] = \lim_{x\to\infty} f(x) \pm \lim_{x\to\infty} g(x)$

(b) $\lim_{x\to\infty} f(x)g(x) = \lim_{x\to\infty} f(x) \cdot \lim_{x\to\infty} g(x)$

(c) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$ if $\lim_{x\to\infty} g(x) \neq 0$.

(d) For any constant k, $\lim_{x\to\infty}[kf(x)] = k \lim_{x\to\infty} f(x)$.

(e) For any positive integer n, (i) $\lim_{x\to\infty}[f(x)]^n = [\lim_{x\to\infty} f(x)]^n$ (ii) $\lim_{x\to\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\infty} f(x)}$

LIMIT OF SOME SPECIAL FUNCTIONS

(i) $\lim_{x\to\infty} \frac{1}{x} = 0$

(ii) $\lim_{x\to-\infty} (1+\frac{1}{x})^x = e$

(iii) $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

(iv) $\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{\tan^{-1} x}{x} = \lim_{x\to 0} \frac{\sin^{-1} x}{x} = \lim_{x\to 0} \frac{\tan^{-1} x}{x}$

(v) $\lim_{x\to\infty} \frac{e^x - 1}{x} = 1$

(vi) $\lim_{x\to\infty} \frac{a^x - 1}{x} = \ln a \;, a > 0$

(v) $\lim_{x\to\infty} \frac{x^n - a^n}{x - a} = na^{n-1}$

INDETERMINATE FORMS $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, 0^0, \infty^0, \infty-\infty, 1^\infty$ resolve indeterminate form before using the limit by using L-hospital rule or by solving the fractions.

DIFFERENTIAL AND INTEGRAL CALCULUS

First Principle: The *derivative* of the function *f(x)* is the function *f'(x)* defined by

$$f'(x) \equiv \frac{d}{dx}\left[f(x)\right] \equiv \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

S.No	Differentiation	Integration
1	$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1}+c$, n≠ -1
2	$\frac{d}{dx}e^{ax} = ae^{ax}$	$\int e^{ax}dx = \frac{e^{ax}}{a}$
3	$\frac{d}{dx}\log_e x = \frac{1}{x}$	$\int \frac{1}{x}dx = \log x$
4	$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$	$\int a^x dx = \frac{a^x}{\log_e a}$
5	$\frac{d}{dx}\sin ax = a\cos ax$	$\int \sin ax\,dx = -\frac{\cos ax}{a}$
6	$\frac{d}{dx}\cos ax = -a\sin ax$	$\int \cos ax\,dx = \frac{\sin ax}{a}$
7	$\frac{d}{dx}\tan ax = a\sec^2 ax$	$\int \tan ax\,dx = \frac{-\log\sec ax}{a} = \frac{\log\cos ax}{a}$ $\int \sec^2 ax\,dx = \frac{\tan ax}{a}$
8	$\frac{d}{dx}\cot ax = -a\cos ec^2 ax$	$\int \cot ax\,dx = \frac{-\log\cos ecax}{a} = \frac{\log\sin ax}{a}$ $\int \cos ec^2 ax\,dx = \frac{-\cot ax}{a}$
9	$\frac{d}{dx}\sec ax\,dx = a\sec ax\tan ax$	$\int \sec ax\tan ax\,dx = \frac{\sec ax}{a}$ $\int \sec x\,dx = \log(\sec x+\tan x) = \log\ \tan(\frac{\pi}{4}+\frac{x}{2})$
10	$\frac{d}{dx}\cos ecax = -a\cos ec\,ax\cot ax$	$\int \cos ecax.\cot ax\ dx = \frac{-\cot ax}{a}$ $\int \cos ec\,x\,dx = \log(\cos ec\,x - \cot x) = \log\tan\frac{x}{2}$
11	$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x$
12	$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = -\cos^{-1}x$
13	$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = \tan^{-1}x$

14	$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = -\cot^{-1}x$
15	$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x$
16	$\frac{d}{dx}\cos ec^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}dx = -\cos ec^{-1}x$
17	**MULTIPLICATION FORMULA** $\frac{\mathrm{d}}{\mathrm{dx}} f_1(x).f_2(x) = f_2(x).\frac{\mathrm{d}}{\mathrm{dx}} f_1(x) + f_1(x).\frac{\mathrm{d}}{\mathrm{dx}} f_2(x)$	**MULTIPLICATION FORMULA** $\int u.v\,dx = u\int v\,dx - \int\{\frac{d}{dx}u\,.\int vdx\}dx$ **Leibnitz′successive integration by Parts** = $u\int vdx - u'\int\int v\,dx^2 + u''\int\int\int vdx^3 \ldots\ldots\ldots\ldots \underbrace{\int\int\int}_{n} vdx^n$
18	**DIVISION FORMULA(Quotient Rule)** $\frac{d}{dx}(\frac{f_1}{f_2}) = \frac{f_2.(\frac{d}{dx}f_1) - f_1.(\frac{d}{dx}f_2)}{(f_2)^2}$	
19	$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{\sqrt{x}}dx = \int x^{-1/2}dx = \frac{x^{1/2}}{1/2}$

Some Other Formulae for Integration

$\int \frac{1}{\sqrt{a^2-x^2}}dx = \frac{1}{a}\sin^{-1}\frac{x}{a}$	$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$
$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log(\frac{a+x}{a-x}) = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$, $-a < x < a$	
$\int \frac{1}{\sqrt{a^2+x^2}}dx = \log(x+\sqrt{a^2+x^2}) = \sinh^{-1}\left(\frac{x}{a}\right)$	$\int \frac{1}{\sqrt{x^2-a^2}}dx = \log(x+\sqrt{x^2-a^2}) = \cosh^{-1}\left(\frac{x}{a}\right)$
$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}]$	
$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}[x\sqrt{x^2+a^2} + a^2\log(x+\sqrt{x^2+a^2})]$	$\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}[x\sqrt{x^2-a^2} + a^2\log(x-\sqrt{x^2+a^2})]$
$\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}[a\sin bx - b\cos bx]$	$\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}[a\cos bx + b\sin bx]$

Differentiation of Hyperbolic Functions:

$f(x)$	$\sinh x$	$\cosh x$	$\tanh x$	$\mathrm{sech}\,x$	$\mathrm{cosech}\,x$	$\coth x$
$\frac{d}{dx}f(x)$	$\cosh x$	$\sinh x$	$\sec^2\mathrm{h}\,x$	$-\tanh x\,\mathrm{sech}\,x$	$-\mathrm{cosech}\,x\coth x$	$\mathrm{cosech}^2\,x$

Note: the order of int. of the function is dependent on the nature of function. For convenience we use the following concept of

ILATE [I-INVERSE FUNCTION; L-LOGRITHEMIC FUNCTION; A-ALGEBRAIC FUNCTION T- TRIGONOMETTREC FUNCTION; E-EXPONENTIAL FUNCTION]

Limit, Continuity, Differentiation and Integration

1. **L'Hospital's Rule:** Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.

   If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ has a finite limit, or if this limit is $+\infty$ or $-\infty$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

   Moreover, this statement is also true in the case of a limit as $x\to a^-, x\to a^+, x\to -\infty,$ or as $x\to +\infty$.

   Working Process::

   Step 1. Check that the limit of $\frac{f(x)}{g(x)}$ is an **indeterminate form** of type $\frac{0}{0}$.

   If it is not, then **L'Hospital's Rule** cannot be used.

   Step 2. Differentiate *f* and *g* separately. [*Note*: **Do not differentiate** $\frac{f(x)}{g(x)}$ **using the quotient rule**]

2. **Chain Rule:** If $h = g\circ f$, i.e. $h(x) = g(f(x))$ and f, g are differentiable, then

   $$h'(x) = g'(f(x))f'(x) = \frac{d}{dx}h(x) = \frac{d}{dx}g\{f(x)\}.\frac{d}{dx}[f(x)].$$

   For example : If $h(x) = \tan e^{\sin x}$ then $h'(x) = \sec^2 e^{\sin x}(e^{\sin x}\cos x)$

3. **HIGHER DERIVATIVES:** If $y = f(x)$ is a function of x, then the nth derivatives of y w.r.t. x is defined as $\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$ if $\frac{d^{n-1}y}{dx^{n-1}}$ is differentiable.

   Symbolically, the nth derivatives of y w.r.t. x is denoted by $y^{(n)}, f^{(n)}(x)$ or $\frac{d^n y}{dx^n}$.

   Remark: 1. $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ but $\frac{d^2y}{dx^2} \neq \frac{1}{\frac{d^2x}{dy^2}}$ But $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

   2. If y is function of z, $z = f(x)$, then $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$ $(\because \frac{dy}{dx} = f'(x)\cdot\frac{dz}{dx})$.

   $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)\frac{dy}{dz} + f'(x)\frac{d}{dx}\left(\frac{dy}{dz}\right) = f''(x)\frac{dy}{dz} + f'(x)\frac{dz}{dx}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)$$

   $$= f''(x)\frac{dy}{dz} + (f'(x))^2\frac{d^2y}{dz^2}$$

4. **Leibniz's Theorem:** Let f and g be two functions with nth derivative. Then

   $$\frac{d^n}{dx^n}[f(x)g(x)] = \sum_{r=0}^{n} C_r^n f^{(r)}(x)g^{(n-r)}(x)$$

   $$= C_0^n f^0(x)g^{n-0}(x) + C_1^n f^1(x)g^{n-1}(x) + C_2^n f^2(x)g^{n-2}(x) + \ldots\ldots\ldots\ldots + C_n^n f^n(x)g^0(x)$$

   where $f^{(0)}(x) = f(x)$.