CORPORATE

INSTITUTE OF SCIENCE AND TECHNOLOGY, BHOPAL

IMPORTANT FORMULA (Engg. Mathematics –III (BE-401))

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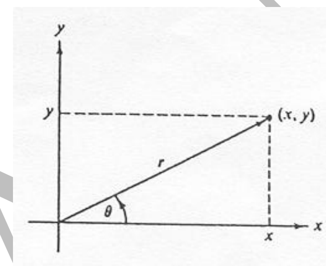
"Education to All for Excellence"

Complex Analysis:

A complex number $z = (x,y) = x + iy$ Where $i = (-1)^{1/2}$. We will see that the ordering of two real numbers (x,y) is significant, *i.e.* in general $x + iy \neq y + ix$.



x is called the real part, labeled by **Re z** and y is called the imaginary part, labeled by **Im z**

For Cartesian components

$$z_1 \pm z_2 = x_1 \pm x_2 + i(y_1 \pm y_2) \text{ and } z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

For polar components, we may write $z = r(\cos\theta + i\sin\theta)$ Or $z = re^{i\theta}$

r = the modulus or magnitude of $z = |z| = \sqrt{x^2 + y^2}$ *and* θ =the argument or phase of z = $\theta = \tan^{-1}\dfrac{y}{x}$

The choice of polar representation or Cartesian representation is a matter of convenience. Addition and subtraction of complex variables are easier in the Cartesian representation. Multiplication, division, powers, roots are easier to handle in polar form, $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$, $z_1 / z_2 = (r_1 / r_2) e^{i(\theta_1-\theta_2)}$, $z^n = r^n e^{in\theta}$

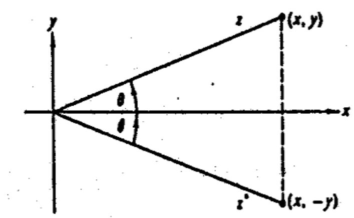
Using the vector analogy, we have the triangle inequalities $|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|$

Using the polar form, $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

From z, complex functions $f(z)$ may be constructed. They can be written $f(z) = u(x,y) + iv(x,y)$ in which v and u are real.

For example if $f(z)=z^2$, we have $f(z) = (x^2 - y^2) + i2xy$

The relationship between z and $f(z)$ is best pictured as a mapping operation, we address it in detail later.



Complex Conjugation: Replacing i by $-i$, which is denoted by , $\bar{z} = x - iy$

We then have $z\bar{z} = x^2 + y^2 = r^2$ hence $|z| = (z\bar{z})^{1/2}$

Note: $z = re^{i\theta}$ or $re^{i(\theta+2n\pi)}$, $\ln z = \ln r + i\theta$ or $\ln z = \ln r + i(\theta + 2n\pi)$

ln z is a multi-valued function. To avoid ambiguity, we usually set n=0 and limit the phase to an interval of length of 2π. The value of *lnz* with n=0 is called the principal value of *lnz*.

Function of Complex Variables: Let $z=x+iy$ is any complex variable , then corresponding function $w=f(z)= u+iv= u(x,y)+iv(x,y)$ is called function of complex variables.

Differentiation of a function: The derivative of $f(z)$, like that of a real function, is defined by

$$f'(z) = \frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

Analytic Function (or Holomorphic function or Regular function or monogenic function):

A function $f(z)$ is said to be **analytic at a point** z_0 , if it is (i) Single Valued and (ii) Differentiable not only at z_0 but at every point of some neighborhood (nbd) of z_0.

A function is said to be **analytic in a domain** if it is analytic at every point of the domain.

- The point at which function is not differentiable is called a **singular point** of the function.
- A function which is analytic at every point of the complex plane is called **Entire function**.
- An Entire function is always analytic , differentiable and continuous but converse is not true.

- A function *f(z)* is said to be meromorphic function in a region R if it is analytic in the region R except a finite number of poles.
- A function *f(z)* is said to be **single valued** if it has only one value for the given value of *z*.
  Example: $f(z)=z^2$ is single valued , $f(z)=z^{1/2}$ is multivalued.
  Note: Every differentiable function may not be analytic but every analytic function will be differentiable.

Cauchy – Riemann equations (Necessary Conditions for a function to be analytic) : If *f(z) = u (x,y) +i v(x,y)* is any function of complex variables ,and *f(z)* is analytic at all the points of the domain (or region) then $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$,provided $\frac{\partial u}{\partial x},\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}$ exist and continuous in the given region.

Cauchy – Riemann equations (Polar form) : If $w=f(r,\theta)$ then $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$

Sufficient condition for *f(z)* to be analytic: The Sufficient condition for *f(z) =u(x,y +iv(x,y)* to be analytic at all the point in a region R

(i) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ (ii) $\frac{\partial u}{\partial x},\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}$ are continuous in R .

- C-R Equation are sufficient condition if the partial derivatives are continuous.

Harmonic Function: A function *f(z)* , Which posses continuous partial derivatives of the first and second order and Whose real or Imaginary part satisfy Laplace equation is called harmonic function.

$$\frac{\partial^2 u}{dx^2}+\frac{\partial^2 u}{dy^2}=0 \quad \text{or} \quad \frac{\partial^2 v}{dx^2}+\frac{\partial^2 v}{dy^2}=0$$

Milne's Thomson Method: (For finding Analytic Function):

1. If Real part *u* of an analytic function *f(z)* is given then $f(z)=\int \phi_1(z,0)dz - i\int \phi_2(z,0)dz + c$

   Where $\phi_1(x,y)=\frac{\partial u}{\partial x}$ and $\phi_2(x,y)=\frac{\partial u}{\partial y}$
2. If Imaginary part *v* of an analytic function *f(z)* is given then $f(z)=\int \phi_1(z,0)dz + i\int \phi_2(z,0)dz + c$

   Where $\phi_1(x,y)=\frac{\partial v}{\partial y}$ and $\phi_2(x,y)=\frac{\partial v}{\partial x}$

Determination of Conjugate Function : (i) If Real part *u* is given then conjugate function *v* can be determine by

$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$,[Using C-R Equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$]

(ii) When Imaginary part v is given, then real part u is determine by $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=\frac{\partial v}{\partial y}dx-\frac{\partial v}{\partial x}dy$

Jordan Arc: A continuous arc without multiple points is called Jordan arc.

- An equation which is satisfied by more than one value of the variable in the given range, then the point is called **multiple points**.

Regular Arc: If the derivatives of the given function are also continuous in the given range, then the arc is called regular arc.

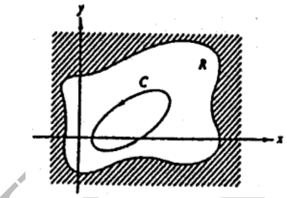
Contour: A contour is a Jordan curve consisting of continuous chain of a finite number of regular arcs.

Zeroes of an analytic function: The Value of *z* for which the analytic function becomes zero is said to be zero of the function.

Cauchy's integral (or Cauchy Goursat)Theorem:

If a function *f(z)* is analytical (therefore single-valued) and its partial derivatives are continuous at **each point with in and on** a simply connected region **R**, for every closed path C in **R**, then $\oint_c f(z)dz = 0$

- **For Multiply connected regions:** $\oint_{C_1'} f(z)dz = \oint_{C_2'} f(z)dz$
- If there is no pole inside and on the contour then the value of the integral of the function is =0



Cauchy's Integral Formula: If f(z) is analytic on and within a closed contour C then $f(z_0) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z - z_0} dz$

Cauchy's Integral Formula for Derivatives: Cauchy's integral formula may be used to obtain an expression for the derivation of f(z) $f'(z_0) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z - z_0)^2} dz$

Moreover, for the n-th order of derivative $f^n(z_0) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z - z_0)^{n+1}} dz$

Zero of an analytic function: A zero of an analytic function *f(z)* is a value of *z* such that *f(z)*=0. *i.e.* An Analytic function is said to have a zero of order *n* if $f(z) = (z-a)^n \phi(z)$

Singularities: Points at which functions are not analytic.

Singular point : z_0 is a **singular point** of *f(z)* ↔ (1) $f(z_0)$ is not analytic. (2) for all $N(z_0)$ there exist $z \in N(z_0)$ in which *f(z)* is analytic.

Isolated singular point: z_0 is an **isolated singular point** of *f (z)* ↔ (1) $f(z_0)$ is singular point (2) for all $N(z_0)$ there exist $f(z) \in N(z_0)$ in which *f(z)* is analytic except z_0. Here $N(z_0)$ denotes the neighborhood of the z_0.

Residue at a pole:

1. **Residue at a simple pole:** If *f(z)* has a simple pole at *z=a* then $\text{Res } f(z)_{\text{at } z=a} = \lim_{z\to a}(z-a) f(z)$
2. **Residue at pole of order *m*:** If *f(z)* has a pole of order *n* at *z=a* [*i.e.* $f(z) = \frac{\phi(z)}{(z-a)^n}$] then

   $\text{Res } f(a)_{\text{at } z=a} = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]$
3. **Residue of f(z) at z=∞** $\text{Res } f(z)_{\text{at } z=\infty} = \lim_{z\to\infty} - z f(z)$

Cauchy's Residues Theorem: If f(z) is an analytic function in a closed curve C , except at a finite number of poles with in C , then $\int_c f(z)dz = 2\pi i(\text{sum of residues at the poles within C})$.

Integration Round the Unit Circle of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)dz$ or $\int_0^{\pi} f(\cos\theta, \sin\theta)dz = \frac{1}{2}\int_0^{2\pi} f(\cos\theta, \sin\theta)dz$:

1. Put $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$ and $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$
2. $z=x+iy = r(\cos\theta+i\sin\theta)=r e^{i\theta}$, *for* unit circle $r=1 \Rightarrow z= e^{i\theta}$, Hence $dz= i e^{i\theta}d\theta$ *or* $d\theta = dz / i e^{i\theta} = dz / i z$
3. Change the given integral $\int_0^{2\pi} f(\cos\theta, \sin\theta)dz = k\int_c f(z)dz$, K is constant term
4. Find the Poles and then residues inside C

5. Use Cauchy's Residues Theorem to evaluate $\int_c f(z)dz = 2\pi i$(sum of residues at the poles within C)

6. Put this value in step -3

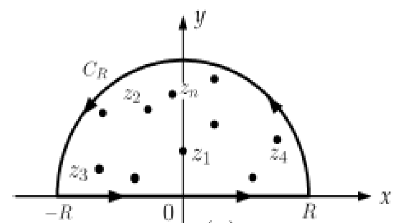
Jordan's lemma: If $\lim_{z\to\infty} f(z) = 0$, uniformly then $\lim_{R\to\infty} \int_{C_R} e^{imz} f(z)dz = 0$,$(m > 0)$

Integration Round the Unit Circle of the type $\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)}dx$ or: $\int_{0}^{\infty} \frac{f_1(x)}{f_2(x)}dx = 2\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)}dx$

Where $f_1(x)$ and $f_2(x)$ are polynomials in x such that $\lim_{x\to\infty} \frac{x\, f_1(x)}{f_2(x)} = 0$,such that $f(x)$ has no zeros on the real axis.

Such types of integrals can be reduced to contour integrals if (i) $f_2(x)$ has no real roots(ii) the degree of $f_2(x)$ is grater than of $f_1(x)$ by at least two.

1. Consider the integral I= $\int_c f(z)dz$, where $f(z) = \frac{f_1(z)}{f_2(z)}$, over the closed contour C , consisting of the real axis from –R to +R and semi circle C_R of radius R in the upper half of the plane.



(a)

2. Then $\int_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx$

3. By Cauchy Residue Theorem

$$\int_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i(\text{sum of residues at the poles within C})$$

4. Solve $\int_{C_R} f(z)dz$: Put $z=Re^{i\theta}$, here R is constant and z moves along C_R and $\theta : 0 \to \pi$, then integral becomes

$$\int_{C_R} f(z)dz = \int_0^{\pi} \frac{f_1(\text{Re}^{i\theta})}{f_2(\text{Re}^{i\theta})}\text{Re}^{i\theta}\, id\theta$$

5. For Large R , $\int_0^{\pi} \frac{f_1(\text{Re}^{i\theta})}{f_2(\text{Re}^{i\theta})}\text{Re}^{i\theta}\, id\theta \to 0, as\, R \to \infty$

6. Put this value in Step-3 , We get $\int_C f(z)dz$ =0+ $\int_{-R}^{R} f(x)dx = 2\pi i$ (Sum of residues in the upper half of the C

7. Find Poles and Residue in the upper half of the C, and put in Step-6.

Bilinear Transformation: A Mobious transformation, or a bilinear transformation, is a rational function

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

Note: If $ad\text{-}bc=0$ then $f'(z) = T'(z) = \frac{ad - bc}{(cz + d)^2}$, becomes zero , then $f(z)$ will be a constant function, which is not possible.

2. We consider f or T defined on C_∞ by defining $f(\infty) = a/c$ (where we interpret a/c as ∞ if $a \neq 0$ and c = 0, and $f(\infty) = \infty$ if $c = 0$) and $f(-d/c) = \infty$ (where we interpret $-d/c$ as ∞ when $c = 0$).

Cross Ratio: If the four points z1 , z2, z3,z4 of Z-plane are taken in order , then the ratio $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$ is called cross ratio.

Invariant or Fixed Point: The points which coincide with their transformation are called invariant points.

Numerical Analysis

Error: (1) Absolute error E_a= Difference between the true value of a quantity and its approximate value=$|x-x_a|$ (2) Relative Error $E_r=|$ Absolute error/True Value$|=|(x-x_a)/x|$ (3) Percentage Error $E_p=(|x-x_a| / x) * 100$

Solution of Algebraic and Transcendental equations:

Algebraic Equation: An equation of the form $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, a_0, a_1, ………..a_n are positive integers and $a_n \neq 0$ is called an algebraic equation of order *n*.

Transcendental Equation: An equation which contains trigonometric, hyperbolic, inverse trigonometric, logarithmic, exponential etc. functions is called Transcendental Equation.

(1) Bisection Method:

(i) Find the negative and positive values of the function at two different points , say *f(a)= -ve* and *f(b)= +ve*

(ii) let $a=x_0$ and $b=x_1$ **(iii)** Find $x_2= x_0 + x_1 /2$ **(iv)** find $f(x_2)$ **(V)** If $f(x_2)$=+ve then root lies b/w $a=x_0$ and x_2

(i) If $f(x_2)$= - ve then root lies b/w $b=x_1$ and x_2 , repeat procedure from (iii)

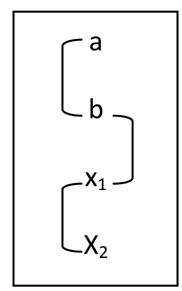
(2) Regula Falsi Method(Or Method of false positions)

(i) Find the negative and positive values of the function at two different points , say *f(a)*= -Ve and *f(b)*= +ve

(ii) let a=x_0 and b=x_1 **(iii)** Find $x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$ **(iv)** Find $f(x_2)$ **(v)** If $f(x_2)$=+ve then root lies b/w $a=x_0$ and x_2 **(VI)**If $f(x_2)$= - ve then root lies b/w b=x_1 and x_2 , repeat procedure from (iii)

Note: This formula is based on linear interpolation. It's rate of convergence is linear. It is slow as compare with secant method.

(3) Secant Method: This method is same as the *Regula Falsi Method*, but in this method we not need to check the +ve and –ve sign in each step. We can use general formula $x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$



Note: (1). This method is fails if $f(x_n) = f(x_{n-1})$.(**2).** This method does not converges always, while Regula falsi method will always converges.**(3).**The only advantage of this method is that , If it converges, then it will converges rapidly than the Regula falsi method.(4) It's rate of convergence is 1.62.

(4) Newton Rap son's Method(Or method of tangent's):

(i) Find the negative and positive values of the function at two different points , say *f(a)*= -Ve and *f(b)*= +ve

(ii) If | *f(a* |< | *f(b)*| then take a= x_0 **(iii)** Find $x_{n+1}= x_n – f(x_n) /f'(x_n)$, Provided $f'(x_n)$ exist **(iv)** find next approximations using (iii)

Note: Newton raphsons' method is useful in cases of large values of *f'(x) i.e.* when the graph of *f(x)* while crossing the X-axis is nearly vertical. This method is also used for finding complex roots. The condition for convergence is $|f(x).f'(x)| < \{f'(x)\}^2$ for all x in the interval in which the root lies.

Rate of Convergence: Bisection Method< Regula Falsi Method< Secant Method< Newton Raphsons Method

(iii) Iterative (Successive Approximation) Method: (1) Find the negative and positive values of the function *f(x)*=0at two different points , say *f(a)*= -Ve and *f(b)*= +ve

(2) Now write the equation $f(x)=0$ in the form $x=\varphi(x)$ such that $|\varphi'(x)|<1$ for all $x\in(a,b)$

(3) Next choose x_0 from (a,b) , say $x_0=(a+b)/2$, then putting this value in $x=\varphi(x)$.

(4) Iterative Values: $x_1=\varphi(x_0)$, $x_2=\varphi(x_1)$, $x_3=\varphi(x_2)$…….

Solution of Algebraic Simultaneous linear equations:

I. Mathematical background

In mathematics, a **matrix** (plural **matrices**) is a rectangular table of numbers or, more generally, a table consisting of abstract quantities that can be added and multiplied.

Example: $\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\Leftrightarrow\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$

The elements in the " middle" of a square matrix form the diagonal of the matrix

Some particular matrices

1. Symmetric matrix: The same numbers are above and below the diagonal

$$\begin{bmatrix}1&2&3\\2&5&6\\3&6&9\end{bmatrix}\Leftrightarrow\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{12}&a_{22}&a_{23}\\a_{13}&a_{23}&a_{33}\end{bmatrix}$$

2. Diagonal matrix: It is a SQUARE matrix where all the elements are zero except on the diagonal: $a_{i,j}=0$ for $i\neq j$ **Example:**

$$\begin{bmatrix}1&0&0\\0&5&0\\0&0&9\end{bmatrix}\Leftrightarrow\begin{bmatrix}a_{11}&0&0\\0&a_{22}&0\\0&0&a_{33}\end{bmatrix}$$

3. Identity matrix: it is a diagonal matrix where all the elements on the diagonal are equal to 1. Noted as [I]

$a_{i,j}=0$ for $i\neq j$ AND $a_{i,i}=1$ $\quad I=[\,I\,]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

4. Upper triangular matrix: It is a matrix where all the elements below the diagonal are equal to zero.
5. Lower triangular matrix: It is a matrix where all the elements above the diagonal are equal to zero.

$$\begin{bmatrix}1&2&3\\0&5&6\\0&0&9\end{bmatrix}\Leftrightarrow\begin{bmatrix}a_{11}&a_{12}&a_{13}\\0&a_{22}&a_{23}\\0&0&a_{33}\end{bmatrix}$$

Lower Triangular Matrix

$$\begin{bmatrix}1&0&0\\2&5&0\\3&6&9\end{bmatrix}\Leftrightarrow\begin{bmatrix}a_{11}&0&0\\a_{12}&a_{22}&0\\a_{13}&a_{23}&a_{33}\end{bmatrix}$$

Upper Triangular matrix

Linear Algebraic Equations: Let system of linear equations is:

$$a_1x+b_1y+c_1z=d_1,\ a_2x+b_2y+c_2z=d_2,\ a_3x+b_3y+c_3z=d_3$$

1. Direct Methods:

(i) Gauss Elimination method (Method of Pivoting) : In essence, we wish to eliminate unknowns from the equations by a sequence of algebraic steps.

Let augmented matrix **[A:b]=** $\left[\begin{array}{ccc|c}a_1&b_1&c_1&d_1\\a_2&b_2&c_2&d_2\\a_3&b_3&c_3&d_3\end{array}\right]$

Normalization (i) Let $a_1\neq0$. Then by $R_{21}(\frac{-a_2}{a_1})\Rightarrow R_2=R_2-\frac{a_2}{a_1}R_1$ and $R_{31}(\frac{-a_3}{a_1})\Rightarrow R_3=R_3-\frac{a_3}{a_1}R_1$

We get $\approx\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & b_3' & c_3' & d_3' \end{array}\right]$ here a1 is called pivoting element.

Reduction : Now take b_2' ($\neq 0$) as the pivoting element , and use $R_{32}\left(\frac{-b_3'}{b_2'}\right) \Rightarrow R_3 = R_3 - \frac{b_3'}{b_2'} R_2$

We get $\approx\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_3'' & d_3'' \end{array}\right]$ after solving this matrix by back substitution we get required results.

Note: This method fails if a_1, b_2' or c_3'' becomes zero. In such cases by inter changing the rows we can get the non zero pivots.

(ii) Gauss Jordan Method: It is a variation of Gauss elimination. The differences are:

- When an unknown is eliminated from an equation, it is also eliminated from all other equations.

- All rows are normalized by dividing them by their pivot element.

Hence, the elimination step results in an identity matrix rather than a triangular matrix. Back substitution is, therefore, not necessary.

All the techniques developed for Gauss elimination are still valid for Gauss-Jordan elimination.

However, Gauss-Jordan requires more computational work than Gauss elimination (approximately 50% more operations).We will do *row operations*:

1. Interchange two rows $R_i \leftrightarrow R_j$
2. Multiply a row by a constant $cR_i \rightarrow R_i$
3. Add a multiple of one row to another $cR_i + R_j \rightarrow R_j$

- on an *augmented matrix* to solve a system
- using a method known as

GAUSS-JORDAN ELIMINATION:

1. Get a 1 in upper left corner (by row ops 1 and/or 2)
2. Get 0's everywhere else in its column (by row op 3)
3. Mentally delete row 1 and column 1. What remains is a smaller **submatrix**.
4. Get 1 in upper lefthand corner of the *submatrix*.
5. Get 0's everywhere else in its column for *all rows* in the matrix (not just the submatrix).
6. Mentally delete row 1 and column 1 of the submatrix, forming an even smaller submatrix.
7. Repeat 4, 5, 6 until you can go no further.
8. The matrix will now be in **reduced row-echelon form** (RREF), or just **reduced form**.
6. Re-write the system in natural form.
7. State the solution.

A. If you get a row of all zeros, use row op 1 to make it the last row

B. If you get a row with all zeros to the left of the line, and a non-zero on the right, STOP (no solution).

Example:

$$\begin{bmatrix} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{bmatrix} R_3 \leftrightarrow R_1 \quad \Rightarrow \quad \begin{bmatrix} \boxed{1} & -3 & 2 & 0 \\ 3 & 1 & -1 & 7 \\ 2 & -2 & 1 & 3 \end{bmatrix} -3R_1+R_2 \rightarrow R_2 ,\ -2R_1+R_3 \rightarrow R_3 \Rightarrow \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & \boxed{10} & -7 & 7 \\ 0 & 4 & -3 & 3 \end{bmatrix} 1/10R_1 \rightarrow R_2$$

$$\Rightarrow\begin{bmatrix}1 & -3 & 2 & 0\\ 0 & \boxed{1} & -\frac{7}{10} & \frac{7}{10}\\ 0 & 4 & -3 & 3\end{bmatrix} 3R_2+R_1\to R_1\ ,\ -4R_1+R_3\to R_3 \quad \Rightarrow\begin{bmatrix}1 & 0 & -\frac{1}{10} & \frac{21}{10}\\ 0 & 1 & -\frac{7}{10} & \frac{7}{10}\\ 0 & 0 & -\frac{1}{5} & \frac{1}{5}\end{bmatrix} 5R_3\to R_3$$

$$\Rightarrow\begin{bmatrix}1 & 0 & -\frac{1}{10} & \frac{21}{10}\\ 0 & 1 & -\frac{7}{10} & \frac{7}{10}\\ 0 & 0 & 1 & -1\end{bmatrix} 1/10R_3+R_1\to R_1\ ,\ 7/10R_3+R_2\to R_2 \quad \Rightarrow\begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}$$

Answer: $x_1 = 2, x_2 = 0, x_3 = -1$

(ii) LU Factorization Method(or Crout's Method , or Choleskey's Method)

For a nonsingular matrix $[A]$ on which one can successfully conduct the Naïve Gauss elimination forward elimination steps, one can always write it as

1. $[A]=[L][U]$ Where : $[L]$= Lower triangular matrix with unit diagonal $= \begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}$

   $[U]$ = Upper triangular matrix= $\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$

2. $[L][Z]=[b]$ 3. $[U][X]=[Z]$ Where $Z=[z_1, z_2, z_3]'$

4. Use back Substitution to find values of *x, y, z*

ITRATIVE METHODS FOR SOLVING SIMULTANEOUS LINEAR EQUATION:

(i) ***Jacobi Method*** **:** Let system of equations is

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1\\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2\\ &\vdots\\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n &= b_n\end{aligned}$$

Solve each equation for one variable:

For first equation $|a_{11}|>|a_{11}|+|a_{12}|.......+|a_{1n}|$, For Second equation $|a_{22}|>|a_{21}|+|a_{23}|.......+|a_{2n}|$ ……

$$\begin{aligned}x_1 &= [b_1 - (a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n)]/a_{11}\\ x_2 &= [b_2 - (a_{21}x1 + a_{23}x_3 + \dots + a_{2n}x_n)]/a_{22}\\ x_3 &= [b_3 - (a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n)]/a_{33}\\ &\vdots\\ x_n &= [b_n - (a_{n1}x_1 + a_{n2}x_2 + \dots + a_{n,n-1}x_{n-1})]/a_{nn}\end{aligned}$$

The Iteration formulas are

$$x_1^{j+1}=\left[b_1-\left(a_{12}\ x_2^j+a_{13}\ x_3^j+\cdots+a_{2n}\ x_n^j\right)\right]\Big/a_{11}$$

$$x_2^{j+1}=\left[b_2-\left(a_{21}\ x_1^j+a_{23}\ x_3^j+\cdots+a_{2n}\ x_n^j\right)\right]\Big/a_{22}$$

$$x_3^{j+1}=\left[b_3-\left(a_{31}\ x_1^j+a_{32}\ x_2^j+\cdots+a_{2n}\ x_n^j\right)\right]\Big/a_{33}$$

……

$$x_n^{j+1}=\left[b_n-\left(a_{n1}\ x_1^j+a_{n2}\ x_2^j+\cdots+a_{n,n-1}\ x_{n-1}^j\right)\right]\Big/a_{nn}$$

(ii) **Gauss-Seidel Method:**
In most cases using the newest values on the right-hand side equations will provide better estimates of the next value. If this is done, then we are using the Gauss-Seidel Method:
The Iteration formulas are:

$$x_1^{j+1} = \left[b_1 - \left(a_{12}\ x_2^j + a_{13}\ x_3^j + \cdots + a_{2n}\ x_n^j \right) \right] \Big/ a_{11}$$

$$x_2^{j+1} = \left[b_2 - \left(a_{21}\ \mathbf{x_1^{j+1}} + a_{23}\ x_3^j + \cdots + a_{2n}\ x_n^j \right) \right] \Big/ a_{22}$$

………………………..

$$x_n^{j+1} = \left[b_n - \left(a_{n1}\ x_1^{j+1} + a_{n2}\ x_2^{j+2} + \cdots + a_{n,n-1}\ x_{n-1}^{j+n} \right) \right] \Big/ a_{nn}$$

Note: 1. ***Why use Jacobi ? Ans:*** Because you can separate the n-equations into n independent tasks; it is very well suited to computers with parallel processors.

2. *If either method converges, Gauss-Seidel converges faster than Jacobi.*

Difference operators:

(1) Shifting Operator : $E f(x) = f(x+h)$, $E^2 f(x) = f(x+2h)$, …….. $E^n f(x) = f(x+nh)$, *or* $E\, y_x = y_{x+h}$, ……. $E^n y_x = y_{x+nh}$,

(2) *Forward difference operator:* $\Delta f(x) = f(x+h) - f(x)$ *or* $\Delta y_x = y_{x+h} - y_x$

(3) *Backward difference operator :* $\nabla f(x) = f(x) - f(x-h)$ *or* $\nabla y_x = y_x - y_{x-h}$

(4) *Averaging operator :* $\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$

(5) *Central difference operator =* $\delta f(x) = f\left(x + \frac{h}{2}\right) - f(x - h/2)$

$\Delta = E - 1$

$\nabla = 1 - E^{-1}$

$\mu = (E^{1/2} + E^{-1/2})/2$

$\delta = E^{1/2} - E^{-1/2}$

Interpolation: Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ……, (x_n, y_n). So, how then does one find the value of y at any other value of x ? Well, a continuous function $f(x)$ may be used to represent the $(n+1)$ data values with $f(x)$ passing through the $(n+1)$ points. Then one can find the value of y at any other value of x. This is called interpolation. Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called **extrapolation.**

Interpolation with equal intervals:

(1) Gregory- Newton's Forward interpolation formula:

$$y = f(x) = y_0 + \frac{p}{1!}\Delta^1 y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots\ldots\ldots + \frac{p(p-1)(p-2)\ldots\ldots(p-(p-1))}{n!}\Delta^n y_0\ ,\ p = \frac{x - x_0}{h}$$

(2) Gregory- Newton's Backward interpolation formula:

$$y = f(x) = y_0 + \frac{p}{1!}\nabla^1 y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \ldots\ldots\ldots + \frac{p(p+1)(p+2)\ldots\ldots(p+(p-1))}{n!}\nabla^n y_0,$$

Central difference interpolation formulas:

(3) Gauss forward difference interpolation formula:

$$y = f(x) = y_0 + \frac{p}{1!}\Delta^1 y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots, 0 < p < 1$$

(4) Gauss Backward difference interpolation formula:

$$y = f(x) = y_0 + \frac{p}{1!}\Delta^1 y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots, 1 < p < 0$$

Gauss Backward	$\Delta\ \mathbf{y_{-1}}$		$\Delta^3\ \mathbf{y_{-2}}$		$\Delta^5\ \mathbf{y_{-3}}$		$\Delta^7\ \mathbf{y_{-4}}$
$\mathbf{y_0}$		$\Delta^2\ \mathbf{y_{-1}}$		$\Delta^4\ \mathbf{y_{-2}}$		$\Delta^6\ \mathbf{y_{-3}}$	
Gauss Forward	$\Delta\ \mathbf{y_0}$		$\Delta^3\ \mathbf{y_{-1}}$		$\Delta^5\ \mathbf{y_{-2}}$		$\Delta^7\ \mathbf{y_{-3}}$

(5) Sterling Formula: { Sterling formula is the mean of gauss forward and back ward formula}

$$y = f(x) = y_0 + \frac{p}{1!}\left[\frac{\Delta^1 y_0 + \Delta^1 y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots, -\frac{1}{4} < p < \frac{1}{4}$$

(6) Bessel's Formula :{Shift the origin to 1 by replacing p by (p-1) & add 1 to each argument 0,-1,-2….in gauss backward formulas and , take mean of gauss forward formula and revised backward formula}

$$y = f(x) = y_0 + \frac{p}{1!}\Delta^1 y_0 + \frac{p(p-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + \frac{p(p-1)(p-1/2)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \ldots\ldots, \frac{1}{4} < p < \frac{3}{4}$$

Interpolation with Unequal intervals:

(7) Divided difference formula:

$$f[x_0,x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad f[x_0,x_1,x_2] = \frac{f[x_2,x_1] - f[x_1,x_0]}{x_2 - x_0}$$

(8) Newton's Divided difference interpolation formula:

$$f(x) = f(x_0) + (x-x_0)\, f[x_0,x_1] + (x-x_0)(x-x_1)\, f[x_0,x_1,x_2] + \ldots\ldots$$

(9) Lagrange's Interpolation formula

$$y = f(x)) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} f(x_1) + \ldots\ldots\ldots\ldots$$

Numerical Differentiation:

(10) Newton's forward difference formula:

$$f'(x) = f'(a+ph) = \frac{1}{h}\left[\Delta f(a) + \frac{2p-1}{2!}\Delta^2 f(a) + \frac{3p^2-6p+2}{3!}\Delta^3 f(a) + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 f(a) + \ldots\ldots\right]$$

$$f''(x) = f''(a+ph) = \frac{1}{h^2}\left[\Delta^2 f(a) + (p-1)\Delta^3 f(a) + (p-1)\Delta^3 f(a) + \frac{6p^2-18p+11}{12}\Delta^4 f(a) + \ldots\ldots\right]$$

(11) Newton's Backward difference formula:

$$f'(x) = f'(x_n+ph) = \frac{1}{h}\left[\nabla f(x_n) + \frac{2p+1}{2!}\nabla^2 f(x_n) + \frac{3p^2+6p+2}{3!}\nabla^3 f(x_n) + \frac{4p^3-18p^2+22p-6}{4!}\nabla^4 f(x_n) + \ldots\ldots\right]$$

$$f''(x) = f''(x_n+ph) = \frac{1}{h^2}\left[\nabla^2 f(x_n) + (p-1)\nabla^3 f(x_n) + (p-1)\nabla^3 f(x_n) + \frac{6p^2-18p+11}{12}\nabla^4 f(x_n) + \ldots\ldots\right]$$

Numerical Integration:

(12) Trapezoidal Rule: $\int_{x_0}^{x_n} y\,dx = h/2\,[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots y_{n-1})]$

(13) Simpson's 1/3 Rule: [4(odd)+2(even)] *[divide the interval in multiple of 2]*

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots\ldots\ldots + y_{n-2})\right]$$

(14) Simpson's 3/8 Rule: [3(1,2,4,5,7……left multiple of 3)+ 2(3,6,9………multiple of 3)]
[divide the interval in multiple of 3]

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + \ldots.. + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \ldots\ldots\ldots\ldots + y_{n-3})\right]$$

(15) Weddle Rule:[1,5,1,6,1,5,1]*[divide the interval in multiple of 6]*

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{10}\left[(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6) + (y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) + \ldots\ldots\right]$$

Numerical Solution of Ordinary Differential Equations:

1. **Picard's Method of Successive Approximation**
   Let us consider the initial-value problem $y'=f(x,y)$, $y(x_0)=y_0$
   then $y_1(x)= y_0 + \int_{x_0}^{x} f(t,y_0)dt$, $y_2(x)=y_0 + \int_{x_0}^{x} f(t,y_1(t))dt$, ...$y_{n+1}(x)= y_0 + \int_{x_0}^{x} f(t,y_n(t))dt$
2. **Euler Methods** Given the initial-value problem $\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$
   Defined on the interval $x_0 \le x \le x_0+h$, then at $x_1=x_0+h$ the **approximate value** of $y(x_0+h)$, denoted by y_1, is given by $y_1=y_0 +h [f(x_0,y_0)]$, $y_2=y_1+h[f(x_1,y_1)]$,..................., $\boldsymbol{y_{n+1}=y_n+h [f(x_n,y_n)]}$
3. **Improved Euler's method**
   The approximate solution $y_n = (y_1, y_2, y_3,.........,y_n)$ is defined by $y_{n+1}=y_n+ h \frac{f(x_n,y_n)+f(x_{n+1},y^*_{n+1})}{2}$
   where $y^*_{n+1} = y_n +h f(x_n,y_n)$
4. **Runge-Kutta Method:**
   $K_1=f(x_0,y_0)$, $K_2=f(x_0+\frac{1}{2}h, y_0 + \frac{1}{2}hK_1)$, $K_3=f(x_0+\frac{1}{2}h, y_0 + \frac{1}{2}h K_2)$, $K_4 =f(x_0+h,y_0+K_3)$
   $y_{n+1}=y_n+\frac{1}{6}h\left(K_1+2K_2+2K_3+K_4\right)$
5. **Milne's Predictor-corrector method**: The third-order equations for predictor and corrector are
   Let differential equation is $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$
   (1) First find four starting values of y values by any previous method(Taylor series, Euler's method, Picard Method , etc.).Find y_0, y_1, y_2, y_3 for x_0 , x_1, x_2, x_3 respectively.
   (2) Find y'_0, y'_1, y'_2, y'_3 (*from* $\frac{dy}{dx} = f(x,y)$)
   (3) Find y_4 using Milne's Predictor formula $y_4 = y_0 + \frac{4h}{3}[2y_1'-y_2'+2y_3']$ and find $y_4'= f(x_4, y_4)$
   (4) Use Milne's Corrector formula and find $y_4^{(1)} = y_2 + \frac{h}{3}[2y_2'+4y_3'+y_4']$ again
   $y_4^{(2)} = y_2 + \frac{h}{3}[2y_2'+4y_3'+y_4'^{(1)}]$, $y_4^{(3)} = y_2 + \frac{h}{3}[2y_2'+4y_3'+y_4'^{(2)}]$ ……
   Continue this process, when two consecutive approximations are equal at desire places

Correlation: In statistics, the word "correlation" has a very specific meaning. Statistical correlation means that, given two variables X and Y measured for each case in a sample, variation in X corresponds (or does not correspond) to variation in Y, and vice versa. That is, extreme values of X are associated with extreme values of Y, and less extreme X values with less extreme Y values. The correlation coefficient (Pearson r) measures the degree of this correspondence.

Correlation and Regression:

Covariance of X and Y : $\text{cov}(X,Y)=\frac{1}{n}\sum_{i=1}^{n}(x-\bar{x})(y-\bar{y})$, Where $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i$

Covariance in terms of data x and y , $\text{cov}(X,Y)=\frac{1}{n}\sum_{i=1}^{n}x_iy_i-\frac{1}{n}(\sum_{i=1}^{n}x_i)(\frac{1}{n}\sum_{i=1}^{n}y_i)$

Karl Pearson's coefficient of correlation: There is a simple and straightforward way to measure correlation between two variables. It is called the Pearson correlation coefficient (r) – named after Karl Pearson who invented it. It's longer name, the Pearson product-moment correlation, is sometimes used.

$$r = \frac{\text{Covarience}(x,y)}{\sqrt{\text{variance } x}\sqrt{\text{variance } y}} = r = \frac{\text{Covarience}(x,y)}{\sigma_x \sigma_y} = \frac{\sum \frac{XY}{n}}{\sqrt{\frac{\sum X^2}{n}\frac{\sum Y^2}{n}}} = \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}} \text{ Where } X = x - \bar{x},\ Y = y - \bar{y}$$

$$r = = \frac{\frac{\sum uv}{n} - \frac{\sum u}{n}\frac{\sum v}{n}}{\sqrt{\frac{\sum u^2}{n} - (\sum \frac{u}{n})^2}\sqrt{\frac{\sum v^2}{n} - (\sum \frac{v}{n})^2}} = = \frac{n\sum uv - \sum u \sum v}{\sqrt{n\sum u^2 - (\sum u)^2}\sqrt{n\sum v^2 - (\sum v)^2}} \text{ where } u = x - A_x \text{ and } v = y - A_y$$

$$r = \frac{1}{n-1}\sum \frac{(x_i - \bar{X})(y_i - \bar{Y})}{s_x s_y}$$

Note : The value of r ranges between +1 and -1.

- r > 0 indicates a positive relationship of X and Y: as one gets larger, the other gets larger.
- r < 0 indicates a negative relationship: as one gets larger, the other gets smaller.
- r = 0 indicates no relationship

2. Spearman's rank correlation coefficient (Rank Correlation):

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)}$$, where d=difference of two ranks, n= no. of observations,

$$r = 1 - \frac{6\left[\sum d^2 + \frac{1}{12}m_1(m_1^2-1) + \frac{1}{12}m_2(m_2^2-1) + \ldots\ldots\right]}{n(n^2-1)}$$,m_i= no of repeated ranks

Regression: "the prediction of one variable from knowledge of one or more other variables"
Linear regression—"regression in which the relationship is linear".
Regression lines :

1. **Regression line of y on x :** $y - \bar{y} = b_{yx}(x - \bar{x})$ or, where b_{yx} = regressions coefficient

2. **Regression line of y on x :** $\tan\theta = \left|\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\right|\left|\frac{\rho^2 - 1}{\rho}\right|$ or $x - \bar{x} = \rho\frac{\sigma_y}{\sigma_x}(y - \bar{y})$,where b_{xy} = regressions coefficient

Regression Coefficients: $b_{xy} = \rho\frac{\sigma_x}{\sigma_y} = \frac{n\sum xy - \sum x \sum y}{n\sum y^2 - \left(\sum y\right)^2}$ and $b_{yx} = \rho\frac{\sigma_y}{\sigma_x} = \frac{n\sum xy - \sum x\sum y}{n\sum x^2 - \left(\sum x\right)^2}$

Properties of Regression Coefficients:

1. Correlation coefficient is the geometric mean of regression coefficients $\rho = \sqrt{b_{xy} \times b_{yx}}$
2. If one of the regression coefficients >1 then other will be <1 *i.e.* $b_{xy} > 1$ then $b_{yx} < 1$
3. The Angle between regression lines θ is given by $\tan\theta = \left|\frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}\right|\left|\frac{\rho^2 - 1}{\rho}\right|$

4. Arithmetic mean of regression coefficient is greater than correlation coefficient $\rho < \frac{b_{xy} + b_{yx}}{2}$

Method of least squares (or Curve Fitting) Method: Method of least squares provides a unique set of values to the constants and hence suggests a curve of best fit to the given data.

1. Fitting a straight line(or Equation of degree one) : Let equation of straight line $y = a+bx$

Then normal equations are: $\sum y = an + b\sum x$, where $\sum 1 = n$ and $\sum xy = a\sum x + b\sum x^2$

2. Fitting a Parabola (or Equation of degree two) : Let equation of parabola is $y = a+bx+cx^2$

Then normal equations are: $\sum y = an + b\sum x$, where $\sum 1 = n$

and $\sum xy = a\sum x + b\sum x^2$ and $\sum x^2 y = a\sum x^2 + b\sum x^3$

On solving normal equations we get the values of a, b, c , and putting these values in equation of straight line or parabola we get the required results.

Basic Concepts Probability

Events and Outcomes

The result of an experiment is called an **outcome**. An **event** is any particular outcome or group of outcomes. A **simple event** is an event that cannot be broken down further. The **sample space** is the set of all possible simple events.

Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event *E* using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event E}}{\text{Total number of equally - likely outcomes}}$$

Cards: A standard deck of 52 playing cards consists of four suits (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different rank: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

Complement of an Event: The complement of an event is the event "*E* doesn't happen".

The notation $\bar{E}$ is used for the complement of event E we can compute the probability of the complement using $P(\bar{E}) = 1 - P(E)$

Independent Events : Events A and B are independent events if the probability of Event B occurring is the same whether or not Event A occurs *P*(*A* and *B*) for independent events

If events *A* and *B* are independent, then the probability of both *A* and *B* occurring is

P(A and B) = P(A) · P(B)

where *P*(*A* and *B*) is the probability of events *A* and *B* both occurring, *P*(*A*) is the probability of event *A* occurring, and *P*(*B*) is the probability of event *B* occurring *P*(*A* or *B*).

The probability of either *A* or *B* occurring (or both) is **P(A or B) = P(A) + P(B) – P(A and B)**

Conditional Probability:

The probability the event *B* occurs, given that event *A* has happened, is represented as $P(B|A)$

This is read as "the probability of *B* given *A*" .

If Events *A* and *B* are not independent, then **P(A and B) = P(A) · P(B | A)**

Bayes' Theorem :

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\bar{A})P(B|\bar{A})}$$

BINOMIAL DISTRIBUTION (or Bernoulli distribution):

A sequence of 'trials' in which: (1) each trial has only two possible outcomes ('success' or 'failure'); (2) the probabilities of success/failure are constant across trials; and (3) trials are statistically independent of each other. (Example: repeatedly flipping a coin and observing 'heads' or 'tails'.)

OR

A random variable X is said to follow binomial distribution with parameters n & p if $P(X) = {}^nC_x\, p^x q^{n-x}$ where x = 0 , 1,2,3 …..n , Where p is the probability of success and q is the probability of failure and $q = 1 - p$ and $p(x)$ is called the probability density function.

The binomial distribution applies in the following situation:

In a binomial distribution : (1) n = The number of trials is finite

(2) each trial has two possible outcomes called success & failure.

(3) There is a constant probability p of success on each trial; *i.e.* p & q is constant for all the trials.

(4) The trials are statistically independent (i.e. the outcome of past trials does not affect subsequent trials);

Probability mass function: If x equals the number of successes in the n trials, we have:

$$P(x=r) = {}^nC_r\, p^r q^{n-r} = \frac{n!}{r!\; n-r!} p^r q^{n-r}$$

Poisson distribution:

Poisson distribution is a discrete probability distribution, which is the limiting case of the binomial distribution under certain conditions.

1. When n is very indefinitely very large
2. Probability of success is very small.
3. $np = \lambda$ is finite, $\lambda \in R^+$

Def: A discrete random variable X is said to be follow a Poisson distribution if the probability mass function is given by

$$p(X = x) = P(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,3..........\infty \quad \text{Where e = 2.7183 and } \lambda > 0$$

Here λ is called the ***parameter*** of the Poisson distribution.

- **Examples where the Poisson distribution is used (or) Applications of Poisson distribution:**

This distribution is used to describe the behavior of the rare events like

1. The number of blind born per year in a large city.
2. The number of printing mistakes per page in a large volume of a book.
3. The number of air pockets in a glass sheet.
4. The number of accidents occurred annually at a busy crossing of city.
5. The number of defective articles produced by a quality machine.
6. This is widely used in waiting lines or queuing problems in management studies.
7. It has wide applications in industrial quality control.
8. In determining the number of deaths in a given period by a rare disease.

For a Poisson distribution the probability mass function is given by

$$p(X = x) = P(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,3..........\infty$$

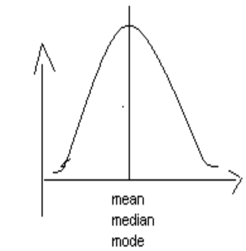
Example: 1) Number of printing mistakes on each page of a book published by a good publisher

2) Number of telephone calls arriving at a telephone switch board per minute.

NORMAL DISTRIBUTION

This is a continuous distribution. It can be derived from the binomial distribution as a limiting case where n The no. of trials is very large.& P the probability of success is close to ½.The general equation is

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{\sigma^2}}$ where the variable x lies between $-\infty < x < \infty$, μ & σ are called the parameters of the distribution $f(x)$ is called probability density function of the normal distribution. The graph of the normal distribution is called the normal curve. It is bell shaped and symmetric about its mean. The two tails of the curve extend to $+\infty$ & $-\infty$ The curve is unimodal. The total area under the curve is 1.



Standard form of normal distribution

If X is a normal random variable with mean μ & standard deviation σ , then the random variable Z = $\underline{X-\mu}$ has the normal distribution with mean 0 and S.D 1.

Then probability density function $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ $.-\infty < z < \infty$ It is free from any parameters.

Beta distribution: It is a continuous distribution.

- It is bounded on both sides. In this respect it resembles the binomial distribution. The standard beta distribution is constrained so that its domain is the interval (0, 1).
- The beta distribution has two parameters *a* and *b* both referred to as shape parameters.
- The formula for the beta density is the following. $f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\Gamma\beta} x^{\alpha-1}(1-x)^{\beta-1} = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$

The reciprocal of the ratio of gamma functions that appears in front as the normalizing constant is generally called the beta function and is denoted *B(α, β)*.

- The beta distribution is often used in conjunction with the binomial distribution particularly in Bayesian models where it plays the role of a prior distribution for *p*.
- It also can be used to give rise to a beta-binomial model. Here the probability of success *p* is assumed to arise from a beta distribution and then, given the value of *p*, the observed number of successes has a binomial distribution with parameters *n* and this value of *p*. The significance of this approach is that it allows *p* to vary randomly between subjects and is a way of modeling what's called binomial over dispersion.

Gamma distribution

- A continuous distribution.
- Like the lognormal the gamma distribution is unbounded on the right, defined for only positive *X*, and tends to yield skewed distributions.
- Like the lognormal, its variance is proportional to the square of the mean. Variance=$g(\mu)= k\,\mu^2$
- Thus the mean-variance relationship cannot be used to distinguish these two distributions.
- It also has two parameters typically referred to as the shape and the scale.
- The Gamma distribution is given by $f(x;\alpha,\beta) = \frac{1}{\Gamma\alpha} x^{\alpha-1}\beta^{\alpha}e^{-\beta x}$ Here $\Gamma\alpha$ is the gamma function, a generalization of the factorial function to continuous arguments. Greek letters are typically used to designate parameters in probability distributions, but it is not uncommon for the parameters of the Gamma distribution to be labeled *a* and *b*.