FIRST-ORDER DIFFERENTIAL EQUATIONS

Method –I : Separation of Variables method This method is used when the equation is in the simplest first-order form of equation e.g. *dy/dx = g(x)/ h(y)* (Basic form)

1. Separate the variables *y* from *x*, i.e., by collecting on one side all terms involving *y* together with *dy*, while all terms involving *x* together with *dx* are put on the other side.
2. Integrate both sides.
3. If the solution can be defined explicitly, i.e., it can be solved for *y* as a function of *x*, then do it. If not, the solution can be defined implicitly, i.e., it cannot be solved for *y* as a function of *x*.

METHOD-II: HOMOGENEOUS ORDINARY DIFFERENTIAL EQUATIONS

A homogeneous ordinary differential equation is an equation of the form *P(x,y)dx+Q(x,y)dy=0* where P and Q are homogeneous of the same order.

Put *y= v x* and *dy/dx = v+ x dv/dx* in the given equation , and use seperation of variavle method.

METHOD-III : LINEAR EQUATION

This method is used when the equation is in the form of $\frac{dy}{dx}+p(x)y=q(x)$ (Basic form)

where *p(x)* and *q(x)* – continuous functions may or may not be constants.

Solution: Find Integral Factor , I.F.= $e^{p(x)dx}$ Then Solution : *y. I.F.* = $\int I.F.Q(x)dx+C$

SECOND AND HIGHER ORDER LINEAR HOMOGENEOUS ODE WITH CONSTANT COEFFICIENTS:

SECOND-ORDER DIFFERENTIAL EQUATIONS $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=r(x)$) , where *p(x)*, *q(x)*, and *r(x)* are continuous functions.

If ***r(x) = 0*** for all *x*, then, the equation is said to be ***homogeneous.***

If ***r(x) ≠ 0*** for all *x*, then, the equation is said to be ***nonhomogeneous.***

Solving Second-Order Linear Homogeneous Differential Equations With Constant Coefficients (When *r(x)* = 0):

Two continuous functions *f* and *g* are said to be *linearly dependent* if one is a constant multiple of the other. If neither is a constant multiple of the other, then they are called *linearly independent*.

Find "*auxiliary equation*" by Replacing $\frac{d^2y}{dx^2}$ with m^2, $\frac{dy}{dx}$ with *m*, and *y* with 1.

Case- I : If *auxiliary equation* has **real and distinct** roots m_1 and m_2 then

Complementary Function , C.F.= $y=y_c(x)=c_1e^{m_1x}+c_2e^{m_2x}$

Case-II : If *auxiliary equation* has real and equal root $m_1 = m_2 = m$ then

C.F. = $y=y_c(x)=c_1e^{mx}+c_2xe^{mx}=(c_1+c_2x)e^{mx}$

Case –III : If *auxiliary equation* has complex roots $m=\alpha\pm\beta i$ (i.e. $m_1=\alpha+\beta i$ and $m_2=\alpha-\beta i$)then

C.F. = $y=y_c(x)=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$

Solution of Second-Order Linear Nonhomogeneous Differential Equations With Constant Coefficients (When $r(x)\neq 0$)

The **General solution** of $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=r(x)$ is $y(x)=y_c(x)+y_p(x)=C.F.+P.I.$

To Find P.I.

1	When $r(x)= e^{ax}$	Put $D= a$, except when $f(a) \neq 0$	$P.I. = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$
2	When $r(x)= e^{ax}$ (Special Case When $f(a)=0$)	Put $D=D+a$ and Solve the equation for $1= x^0$ or e^{0x}	$P.I. = \frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(D+a)}1$
3	When $r(x)= Sinax$ or $Cosax$	Put $D^2=-a^2$ and Solve the equation for D by rationalization of the equation (same for *cosax*) Except $f(-a^2) \neq 0$	$P.I. = \frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$
4	When $r(x)= Sinax$ or $Cosax$ (Special Case)	If $f(-a^2) = 0$ *then use*	$P.I. = \frac{1}{f(D^2)}\sin ax = \frac{x}{2a}\cos ax$ $P.I. = \frac{1}{f(D^2)}\cos ax = -\frac{x}{2a}\sin ax$
5	When $r(x)= x^m$	Expand Series $f(D)^{-1}$ using $(1-x)^{-1}= 1+x+x^2+x^3......$ $(1+x)^{-1}=1-x+x^2-x^3......$	$P.I. = \frac{1}{f(D)}x^m = f(D)^{-1}x^m$
6	When $r(x)= e^{ax}V$ (Where V is the function of x)	Put $D= D+a$ for e^{ax} and then use given formua (For solving V use formula from 1 to 5)	$P.I. = \frac{1}{f(D)}x.e^{ax} = e^{ax}\frac{1}{f(D+a)}V$
7	When $r(x)= x V$ (Where V is the function of x)	For solving V use formula from 1 to 5	$P.I. = \frac{1}{f(D)}x.V = x\frac{f'(D)}{[f(D)]^2}V$

Second Order Diffrential Equation with Variable coefficent (or Linear diff. Equation of second order:

The Standerd form of *Second Order Diffrential Equation with Variable coeffecent is*

$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$, *Where P(x), Q(x) and R(x)* are function of x alone or may be constants.

Methods for Solution:

1. **To find One integral of C.F. by inspection: Method-I : If $1+P/a+Q/a^2=0$ then $y= e^{ax}$ *is* one part of the solution (C.F.).**

 If $1+P+Q=0$ then $y= e^x$ *is* one part of the solution (C.F.).

 If $1-P+Q=0$ then $y= e^{-x}$ *is* one part of the solution (C.F.).

Method-I : If $m(m-1)+Pmx+Qx^2=0$ then $y= x^m$ is one part of the solution (C.F.).

 If $P+Qx=0$ then $y=x$ is one part of the solution.

 If $2+2Px+Qx^2=0$ then $y= x^2$ is one part of the solution (C.F.).

Steps for solution:

1. Find One part of the C.F. using formula given above
2. Let $y= u.v$ is complete solution (Where u is the one part of the solution).
3. Substitute the values of y , y' and y'' in given eq..
4. Now we get Second order diff. eq. with constant coefficients in terms of v.
5. Solve this reduced equation for v and find the value of Complete solution $y=u.v$.

2. **Method –II : Removal of First Derivative [or Change of Dependent Variable]**
Steps for solution:

1. Change the given equation in standerd form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$, and find *P(x), Q(x) and R(x).*
2. Let $u=e^{\frac{-1}{2}\int P dx}$
3. Let *y*= *u.v* is complete solution , where *u* is the one part of the solution , finding in step-2
4. Subsitute the value of *y , y'* and *y''* in given equation , We get Normal equation
$\frac{d^2v}{dx^2}+Iv=R/u$, Where $I=Q-\frac{1}{2}\frac{dp}{dx}-\frac{1}{4}P^2$

Method-III : Change of Independent Variable:

1. Change the given equation in standerd form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$,and find *P(x), Q(x) and R(x).*
2. Choose z such that $Q_1=\frac{Q}{(\frac{dz}{dx})^2}=c^2=1$
3. Find $P_1=\frac{[\frac{d^2z}{dx^2}+P\frac{dz}{dx}]}{(\frac{dz}{dx})^2}$ and $R_1=\frac{R}{(\frac{dz}{dx})^2}$
4. Substitute these values in Transformed equation $\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+Q_1(x)y=R_1(x)$
5. Solve this transformed equation by previous methods.

3. **Method of variation of Parameters:** This method is used to find the complete solution, without finding the particular solution.
   1. Find the C.F. of the equation
   2. Let Complete solution is *y= A u(x) + B v(x),* where *u(x)* and *v(x)* are the parts of *C.F.*
   Where A and B can be determined by A=$\int \frac{R\ u}{\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}}\ dx$ +c$_1$ and A= $-\int \frac{R\ v}{\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}}\ dx$ +c$_2$

Series Solution of Differential equations:

1. **When *x*=0 is an ordinary point (i.e. at *x*=0 function is analytic)**
Step-I *let Series solution is* $y=\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n + \ldots$
Step-II find *y , dy/dx,* d^2y/dx^2 and put in question
Step-III Equate to 0 , the coefficient of x^0, x^1, x^2, …….and solve these relations, and get the values of constants $a_1, a_2, \ldots\ldots a_n$, and then find the general solution by equation of step 1.
2. **When *x*=0 is a Regular singular point *i.e.* not an ordinary point (*i.e.* at *x*=0 function is not analytic)[Frobenius Method]**
Step-I *let Series solution is* $y=\sum_{r=0}^{\infty} a_r x^{m+r} = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n + \ldots$
Step-II find *y , dy/dx,* d^2y/dx^2 and put in question.

Step-III Put the lowest power of *x* =0 and find the *indicial equation* , and find the roots of indicial equation (i.e. different values of *m)*
Step-IV If
(i) Roots of indicial equation are ***distinct and not differ by an integer*** then Complete solution
$$y=c_1(y)_{m=m_1}+c_2(y)_{m=m_2}$$

(ii) Roots of indicial equation are equal (*i.e.* $m_1=m_2$) then Complete solution

$$y = c_1(y)_{m=m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m=m_1}$$

(iii) Roots of indicial equation are ***distinct and differ by an integer*** (*i.e.* m_1-m_2= an integer)
Then there aries two cases :

(a) If $m_1<m_2$ and some of the coefficient of y series becomes infinite at $m=m_1$. Then replace a_0 by $(m-m_1)A_0$, where $A_0 \neq 0$ is a constant

Then Complete solution is given by $$y = A_0(y)_{m=m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m=m_2}$$

(b) If $m_1<m_2$ and some of the coefficient of y series becomes indeterminate when $m=m_1$. The complete solution is given by on putting $m=m_1$ in y which contains two arbitrary constants . If we put $m=m_2$ in y then we obtain a series which is a constant multiple of one of the two series contained in the first solution.

Classification of Differential Equations for finding their solutions: